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ON THE SELFOSCILLATORY MODES OF MOTION OF A GAS IN PIPES*

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One-dimensional, non-linear selfexcited oscillations of an ideal gas in pipes are studied. One end of the pipe is closed, and boundary conditions connecting in prescribed manner the incident and reflected Riemann invariants are specified at the other end. Periodic solutions containing shock waves are constructed. A relation connecting the amplitude and the period of the oscillatory motion of the gas is established. The solutions obtained are analysed numerically for stability. The investigations are based mainly on the results of /1-6/ where the forced resonant and subresonant oscillations of a gas in open a closed pipes were studied.

In /1-4/ the equations of oscillations were derived using a method analogous to the Poincaré-Lighthill method of deformed coordinates. The problem was reduced to finding the solutions of ordinary differential equations on the smooth segments, followed by the introduction of discontinuities based on special additional assumptions. In /5, 6/ a sequential approach to solving the class of problems in question was described, within whose framework the problem of discontinuities was solved correctly by analysing the evolution of the compression wave.

The formulation of the boundary value problems in the present paper is related, to a known degree, to the analogous formulations in the investigations of motion of a gas in a Hartman generator where the flows are also oscillatory**. (**A review of such investigations is given in: Dulov V.G. and Maksimov V.P. Thermoacoustics of semiclosed volumes. Preprint 28-86, Novosibirsk, Inst. Theoretical and Applied Mechanics, Siberian Section, Academy of Sciences of the USSR, 1986.). We will use the approach developed in /5, 6/ to analyse the oscillations. The cumbersome derivations given in these papers will be omitted. The arguments concerning the applicability of the isentropic approximation and the possibility of neglecting the change in the Riemann invariants when the characteristics interact with the shock waves, also retain their validity in the case of the oscillations investigated here.

1. Equations of motion.

The equations of gas dynamics in their characteristic form and in the commonly accepted notation are /7/

$$\left(\frac{\partial u}{\partial t}\right)_\xi+\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\xi=0,\quad \left(\frac{\partial u}{\partial t}\right)_\eta-\frac{1}{\rho a}\left(\frac{\partial p}{\partial t}\right)_\eta=0,\quad \left(\frac{\partial s}{\partial t}\right)_\zeta=0$$

where the following operators of differentiation along the characteristics C^+, C^-, C° are used:

$$\left(\frac{\partial}{\partial t}\right)_\xi=\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x},\quad \left(\frac{\partial}{\partial t}\right)_\eta=\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x},$$
$$\left(\frac{\partial}{\partial t}\right)_\zeta=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$$

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We shall consider the small amplitude waves ε. As we said before, we can neglect the change in entropy in the shock waves, and regard the flow as isentropic.

The equations of motion are reduced to the form ($J^\pm$ are the Riemann invariants)

$$\left(\frac{\partial J^+}{\partial t}\right)_\xi = 0, \quad \left(\frac{\partial J^-}{\partial t}\right)_\eta = 0 \quad \left(J^\pm = u \pm \frac{2a}{\varkappa - 1}\right)$$

We shall consider two types of boundary conditions. Let the right-hand end of the pipe (of length X) be closed $u(X, t) = 0$, and let a linear relation exist at its left-hand end between the perturbations of the incident and reflected Riemann invariant (henceforth we shall denote the parameters of the unperturbed gas by a zero subscript)

$$J^+(0, t) - J_0^+ = \nu(1 + \delta)[J^-(0, t) - J_0^-] \tag{1.1}$$
$$\nu = \pm 1, \ \delta \ll 1$$

When $\nu = -1$ (case 1), the boundary condition formulated above reduces to the form

$$u(0, t) = -\delta[J^-(0, t) - J_0^-]/2$$

This is an analogue of the problem of the oscillations of a piston /1, 2, 5/ in which the shock wave in the flow survives a collision with the boundary. If $\nu = +1$ (case 2), then (1.1) can be rewritten thus:

$$p(0, t) - p_0 = 2\delta[J^-(0, t) - J_0^-]/\varkappa$$

This formulation is similar to the way of specifying the pressure /3, 6/. Here the shock wave is reflected from the boundary locally, in the form of a centered rarefaction wave.

The boundary conditions (1.1) represent a special case of a wide class of the linear conditions of reflection, connecting the perturbations arising in the incident and reflected Riemann invariants, with entropy. They are characteristic for the formulations of the problems in studies of the stability of, for example, detonation waves /8, 9/ or flows through nozzles /10-12/. The shock and detonation waves, the Jouguet plane, the cross-section of the nozzle exist, etc., can serve as the flow boundaries.

In the present case conditions (1.1) can be regarded as the expression of the interaction of the flow within the pipe, with the external flow or with some construction. They are of a purely model character.

Let us introduce the dimensionless coordinates using the formulas

$$p = p_0(1 + \varepsilon p'), \ a = a_0(1 + \varepsilon a'), \ u = a_0 \varepsilon u'$$
$$J^\pm = a_0\left(\varepsilon J^{\pm\prime} \pm \frac{2}{\varkappa - 1}\right), \quad t = Tt', \quad x = a_0 T x'$$

In what follows, we shall omit the primes accompanying the dimensionless variables.

In the new variables the equations of motion and the boundary conditions become

$$\left(\frac{\partial J^+}{\partial t}\right)_\xi = 0, \quad \left(\frac{\partial J^-}{\partial t}\right)_\eta = 0, \quad \left(\frac{\partial x}{\partial t}\right)_\xi = 1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi) +$$
$$\frac{3-\varkappa}{4}\varepsilon J^-(\eta), \quad \left(\frac{\partial x}{\partial t}\right)_\eta = -1 + \frac{\varkappa+1}{4}\varepsilon J^-(\eta) +$$
$$\frac{3-\varkappa}{4}\varepsilon J^+(\xi)$$
$$J^+(n, t) + J^-(n, t) = 0, \quad J^+(0, t) = \nu(1+\delta)J^-(0, t), \quad n = X/(a_0 T)$$

We shall associate the characteristic variable $\xi(\eta)$ with the instant of emergence of the corresponding characteristic $C^+(C^-)$ from the left (right) boundary. Then, integrating the equations for the characteristics we obtain

$$C^+: x = \left[1 + \frac{\varkappa+1}{4}\varepsilon J^+(\xi)\right](t - \xi) + \frac{3-\varkappa}{4} I^+$$
$$C^-: x = n - \left[1 - \frac{\varkappa+1}{4} J^-(\eta)\right](t - \eta) + \frac{3-\varkappa}{4} I^-$$
$$I^+ = \int_\xi^t J^-(\eta)\, dt, \quad I^- = \int_\eta^t J^+(\xi)\, dt$$

The integrals in the expressions for $I^\pm$ are taken along the characteristics $\xi, \eta = \text{const}$. In accordance with the arguments of /5, 6/, the position of the characteristics must be determined with an accuracy of order of ε and ε^2 for cases 1 and 2 respectively. We shall seek periodic solutions of the problem of oscillations, assuming that $\delta \ll 1$.

We shall evaluate the integrals $I^{\pm}$ using the method of successive approximations, with the characteristics $x = t - \xi$, $x = -t + \eta + n$ of the unperturbed gas as the first approximation. It is clear that the integrals

$$I_0^+ = \int_{\xi}^{t} J^-(2\tau - \xi - n)\, d\tau, \quad I_0^- = \int_{\eta}^{t} J^+(2\tau - \eta - n)\, d\tau$$

on the piecewise smooth solutions differ from the exact values of $I^{\pm}$ by a quantity of order $O(\varepsilon)$. From this it follows that the equations of the characteristics in which $I^{\pm}$ have been replaced by $I_0^{\pm}$, describe in the xt plane the curves deviating from the true characteristics of the flow by an amount of the order of $O(\varepsilon^2)$.

We will use the equations of characteristics in the first approximation to find the next approximation to $I^{\pm}$.

After lengthy reduction /5, 6/ we obtain a formulas for determining the time at which the characteristic O^+ will return to the left boundary after reflection from the right wall:

$$t_2 = \xi + 2n\left[1 - \frac{\varkappa + 1}{4}\varepsilon J(\xi)\right] \tag{1.2}$$

for case 1, and

$$t_2 = 2n\left[1 - \frac{\varkappa + 1}{4}\varepsilon J(\xi)\right] + \varepsilon^2 \alpha J^2(\xi) + \varepsilon^2 \beta + \xi \tag{1.3}$$

$$\alpha = n\frac{(3\varkappa + 7)(\varkappa + 1)}{32}, \quad \beta = \frac{(3 - \varkappa)(13 - 3\varkappa)}{64}\int_{\xi}^{t_2} J^2\, d\xi$$

for case 2.

The above relations together with the constraints

$$J(t_2) = J(\xi)(1 + \delta) \tag{1.4}$$

$$J(t_2) = -J(\xi)(1 + \delta) \tag{1.5}$$

which follow from the boundary conditions, form a closed system of equations determining the solution of the problem. Here and henceforth the plus sign in J^+ will be omitted. We note here that in deriving (1.2) and (1.3) we have taken into account the possibility of the intersection of the corresponding characteristic with the shock waves of the opposite family /5, 6/.

In the linear approximation, (1.2)-(1.5) have solutions of the form

$$J(t) = e^{i(k/n)\pi t} \cdot e^{\theta t}, \quad \theta = (2n)^{-1}\ln(1 + \delta) \tag{1.6}$$

$k = m$ for $\nu = -1$, $k = m + 1/2$ for $\nu = +1$, m is an integer. When $\delta < 0$ the solutions decay, while when $\delta > 0$ they increase exponentially with time. In the latter case the state of rest of the gas is unstable under arbitrarily small perturbations which increase in the first approximation, without limit. Actually, after sufficiently long periods the non-linear effects begin to manifest themselves, and this may, in general, lead to stabilization of the solution. This problem will be discussed below.

In deriving Eqs.(1.2) and (1.3) we assumed implicitly that the mean values of all quantities averaged over a period are zero. This imposes restrictions on the possible type of perturbations, namely they must not, in the mean, cause the gas to depart from its unperturbed state. This assumption can, in principle, be removed by reformulating the boundary conditions in an obvious manner.

2. Investigation of the equations of oscillations. We shall seek periodic solutions of problems (1.2), (1.4) and (1.3), (1.5) with period $M = kn_0$, $n_0 = n + \Delta$, $\Delta \ll 1$, $k = 2$ for case 1 and $k = 4$ for case 2.

We shall consider each case separately.

Expanding $J(t_2)$ in a Taylor series and using the condition of periodicity, we obtain from (1.2) and (1.4) an ordinary differential equation which is satisfied by the solution sought in the intervals of smoothness

$$[2\Delta - 1/2 n(\varkappa + 1)\varepsilon J(\xi)]\, dJ/d\xi = \delta J(\xi) \tag{2.1}$$

This at once yields the constraints: $\varepsilon \sim \delta \sim \Delta$.

The equation is easily integrable, and when $\Delta = 0$, its solution, apart from the trivial case $J \equiv 0$, is a linear function

$$J = 2\delta(\xi + C)/(n \cdot \varepsilon(\varkappa + 1))$$

Passing now to case 2, we will use an example given in /5, 6/. From (1.3) it follows that the instant t_2 in which the characteristic C^+ returns to the left boundary after passing the pipe twice in the forward and reverse direction, is given, to within terms of higher order of smallness, by the formula

$$t_2 = \xi + 4nk + 2\varepsilon^2 [\alpha J^2(\xi) + \beta]$$

The corresponding value of the invariant J is, by virtue of the boundary conditions,

$$J(t_2) = J(\xi)(1+\delta)^2$$

Expanding the function $J(t_2)$ in a Taylor series in the neighbourhood of $t_{20} = \xi + 4n_0$ we arrive, by virtue of the assumption of its periodicity, at the differential equation

$$[4\Delta + 2\varepsilon^2\alpha J^2(\xi) + 2\varepsilon^2\beta]\, dJ/d\xi = 2\delta J(\xi) \tag{2.2}$$

from which we obtain the estimates $\varepsilon^2 \sim \delta \sim \Delta$.

Eq.(2.2) can also be integrated simply to give

$$(2\Delta + \varepsilon^2\beta)\ln|J| + \varepsilon^2\alpha J^2/2 = \delta t + C$$

The resulting transcendental equation can be used to obtain J as a function of t. Now, knowing Δ, the constants of integration, and the positions of the discontinuities, we can use the above integrals to construct the solutions of the problem of oscillations, which will be continuous or discontinuous. When such an approach is used /1-4/, the rules by which we choose the constants of integration and introduce strong discontinuities into the solution, become important. Moreover, a question arises in the problem of selfexcited ocillations of determining the period of oscillations, or, which is the same, the quantity Δ. All these problems cannot be solved without making some special assumptions.

In what follows, we shall follow the routes taken in /5, 6/. We recall that these papers dealt with a wide class of problems in which the problems of non-linear periodic oscillations of a gas in pipes appear as a special case. The successive asymptotic analysis of the equations of motion made it possible to justify correctly the method of introducing strong discontinuities into the flow. At the same time, an algorithm was given for constructing the solution by the method of evolution. The method is based on the fact that the differential equations of the characteristics can, by virtue of the special character of the flow, be integrated, and their coordinates (x, t) found at a finite distance from their origin (Eqs.(1.2) and (1.3)).

The shock waves form in the flow as a result of the intersection of the characteristics of the same family. This corresponds to the appearance of multivaluedness in the profile of the corresponding invariant, e.g. in the case of dependence on t for fixed x. A discontinuity in the region where the solution is multivalued is introduced from the condition that the areas bounded by the curve $J(x, t)$ lying on the opposite sides of the shock wave, are equal /7/.

The use of this rule for problems of forced oscillations is substantiated in /5, 6/. All arguments appearing in these papers are applied unchanged to the class of problems discussed in the present paper. Relations (1.2)-(1.5) and the rule formulated above governing the introduction of discontinuities into the flow, based on the assumptions made in Sect.1 about the character of the perturbations, lead to the integral law of conservation of momentum.

$$\int_0^{kn_0} J(\xi)\, d\xi = 0$$

3. Numerical results. The periodic solutions of (1.2)-(1.5) were constructed according to the scheme given in /5, 6/. First, a distribution $J = J_0(\xi)$ corresponding to the conditions

$$\int_{\xi_0}^{\xi_0+L} J(\xi)\, d\xi = 0, \qquad J(\xi_0) = J(\xi_0 + L) \tag{3.1}$$

was specified on the segment $[\xi_0, \xi_0 + L]$ ($L = kn$ (for cases 1 and 2 respectively). The initial segment was then transformed by (1.2) and (1.3) into $[t(\xi_0), t(\xi_0) + L]$. If the solution became "inverted" in the course of this process, then the shock waves were introduced into the regions of multivaluedness using the area rule. The values of $J(\xi)$ and $[t(\xi_0), t(\xi_0) + L]$ were found using formulas (1.4) and (1.5). It is clear that the new function constructed on $[t(\xi_0), t(\xi_0) + L]$ also satisfies (3.1). The new values of $J(\xi)$ were continued periodically to $[\xi_0, \xi_0 + L]$, and the process was repeated until the solution was obtained.

An additional modification was introduced into the above computational scheme. The modification was justified by the fact that in the problems discussed up to now, the period of forced oscillations was known in advance, while here it was determined in the course of constructing the solution. The convenience of realizing the computational algorithm was the factor in favour of solving the inverse problem: the period $T = kn_0$ was specified as constant, and the length of the pipe n was chosen during the computation so that the numerical solution would be of period T.

The process of correcting the pipe length was carried out as follows. Let the magnitude n_i from the previous iteration and the coordinate of some point t_i of the solution (such as a point at the shock front or one of the zeros of the function) be both known. Now let this point have, as a result of the above procedure of transforming the profile of J and its periodic continuation, a new coordinate t_f. The length of the pipe for carrying out the next iteration was assumed to be equal to

$$n_f = n_i - \min\{x_f - x_i,\ x_f - x_i + 2T,\ x_f - x_i - 2T\}/k$$

The initial approximation to n was chosen to be equal to n_0. The meaning of such a correction process becomes clear by considering formulas (1.2) and (1.3).

Fig.1 shows the results of the computations for case 1. Here $\varepsilon = \delta = 10^{-1}$, $n_0 = \pi$.

The solid line represents the result in the case when the function

$$J_0(\xi) = -4(\varkappa + 1)^{-1}\sin\xi \tag{3.2}$$

is used as the initial function.

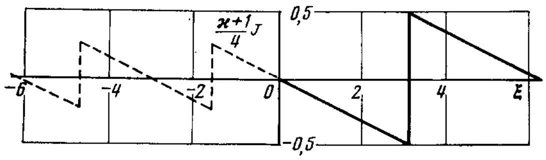

Fig.1

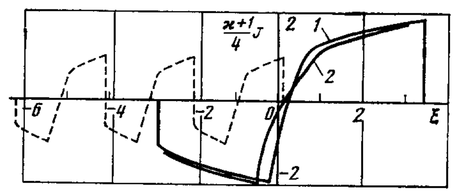

Fig.2

The calculations have shown that $\Delta = 0$. In accordance with the analytic investigations carried out in Sect.2, the solution sought consists of two straight lines over a period, connected by a shock front, or in other words, it represents a symmetrical N-wave. (We shall use this term to characterize the resulting profiles, although in hydrodynamics the term usually refers to a solution consisting of two shock waves connected by a straight line section).

The dashed line shows the result of the computation when

$$J_0 = -4(\varkappa + 1)^{-1}\sin 2\xi$$

We see that here we have two symmetrical N-waves, which could be expected from the arguments of the linear theory (1.6).

Generally speaking, we have found that an arbitrary initial distribution $J_0(\xi)$ yields a solution which can be represented on the segments $(\varphi + 2\pi m,\ \varphi + 2\pi(m+1))$ (φ is the phase determined by J_0) in the form of a collection of independent, symmetrical N-waves separated by segments at rest. However, numerical computations have shown that these solutions are unstable under small perturbations. When such perturbations are introduced artificially, they merge to give a single N-wave (the solid line in Fig.1). The latter wave is stable under the perturbations which result in a phase shift of the solution. A solution with a single N-wave was discussed in /13/ in connection with the study of nearly resonant oscillations in a moving gas.

Fig.2 shows the result of computations for case 2 with $\delta = \varepsilon^2, n_0 = \pi/2, \varepsilon = 0{,}1$ (curve *1*) and $\varepsilon = 0{,}2$ (curve *2*). We used the same function (3.2) to describe the initial distribution for 1 and 2, and the graphs are shown in a single phase. We see that both solutions follow each other closely everywhere except in the neighbourhood of the compression wave whose continuation for curve *2* is, according to Eq.(1.3), about twice as long as that for curve *1*. Unlike case 1, here the period of oscillations depends on the amplitude of the solution. Computations have shown that $\Delta = 0{,}866\varepsilon^2\beta, \beta = 4{,}71$ for solution 1; $\Delta = 0{,}855\varepsilon^2\beta, \beta = 4{,}30$ for solution 2. The conclusions concerning the nearness of the solutions for different ε agree with the laws of similitude obtained as a result of the analytic investigation carried out in Sect.2.

Computations were carried out, aimed at constructing non-linear analogues of (1.6) with higher-order harmonics. The results are shown in Fig.2 for the initial function $J_0 = -4(\varkappa + 1)^{-1}\,\mathrm{sim}\,3\xi$, $\varepsilon = 0{,}1$ with a dashed line ($\Delta = 0{,}9\varepsilon^2\beta$, $\beta = 1{,}3$). The solution can be regarded, with high accuracy, as stationary over a large number of iterations (the number increases as the number of nodes increases when the discrete approximation to equations (1.3) and (1.5) is used in the computational algorithm for the digital computer), but in the end it decomposes and transforms into the one represented by the solid line *1* (apart from the phase shift). We note that the instability of the high-frequency modes relative to the computational algorithm also occurs in case 1.

It would appear that the solutions obtained can be examined vigorously for stability. It is, however, quite clear, e.g. by inspecting the dashed curve of Fig.1, that the violation of symmetry of the N-waves in the case when condition

$$\int_0^{2n_0} J(\xi)\,d\xi = 0$$

holds, must lead to a merger of two shock waves into a single shock wave, and this means the passage to a mode depicted in Fig.1 by the solid line.

The low-frequency solution shown in Fig.2 by the solid lines is stable with respect to the computational scheme, as well as under small perturbations. The latter lead to a phase shift in these curves, and the shift is smaller the smaller the perturbation amplitude.

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